A DIRECT AND SIMPLE PROOF OF JACOBI IDENTITIES FOR DETERMINANTS

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Abstract. The Jacobi identities play an important role in constructing the explicit exact solutions of a broad class of integrable systems in soliton theory. In the paper, a direct and simple proof of the Jacobi identities for determinants is presented by employing the Plücker relations.

1. Introduction

Let $A = \big(a_{ij}\big)_{n\times n}$ be a n-order matrix. Denoted by M_{ij} $\big(R_{ij} \equiv (-1)^{i+j} M_{ij}\big)$ the cofactor (algebraic cofactor) of the matrix entry a_{ij} . The cofactor (algebraic cofactor) of the minor determinant $\begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix}$ is denoted by $M^{i\,j}_{k\,l}$ $\big(R^{ij}_{k\,l} \equiv (-1)^{k+l+i+j} M^{i\,j}_{k\,l}\big)$, then the following Jacobi identities [1, 2]

$$M_{ii}M_{jj} - M_{ij}M_{ji} = M^{i\,j}_{i\,j}\det A, \quad 1 \le i,\, j \le n, \tag{1}$$

are valid. Though the Jacobi identities have been proved in [1], as the author in [2] said, looking at the proof of the general case, it is difficult to understand the Jacobi identities immediately and the author himself came to understand the result by checking the formulae using computer algebra, looking for an alternative proof and applying it to actual problems. Here we will present a direct proof for the Jacobi identities using the famous Plücker relations for determinants.

2. Plücker relations

In this section, let's state the Plücker relations for determinants.

Theorem 2.1. *Let M be a $n\times(n-r)$ matrix and $a_1,\, a_2,\, \cdots,\, a_{2r}$ $2r$ n-order column vectors, then*

$$\sum_\sigma (-1)^{k_1+\cdots+k_r} \begin{vmatrix} M & a_{k_1} & \cdots & a_{k_r} \end{vmatrix} \cdot \begin{vmatrix} M & a_{k_{r+2}} & \cdots & a_{k_{2r}} \end{vmatrix} = 0, \tag{2}$$

where $\big\{k_1,\, k_2,\, \cdots,\, k_{2r}\big\}$ is a permutation of $\big\{1,\, 2,\, \cdots,\, 2r\big\}$ and σ is the permutation with $1 \le k_1 < \cdots < k_r \le 2r$ and $k_{r+1} < \cdots < k_{2r}$.

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Proof. Firstly, it is obvious that

$$
\begin{aligned}
\begin{vmatrix} M & 0 & a_1 & a_2 & \cdots & a_{2r} \\ 0 & M & a_1 & a_2 & \cdots & a_{2r} \end{vmatrix} &= \begin{vmatrix} M & -M & 0 & 0 & \cdots & 0 \\ 0 & M & a_1 & a_2 & \cdots & a_{2r} \end{vmatrix} \\
&= \begin{vmatrix} M & 0 & 0 & 0 & \cdots & 0 \\ 0 & M & a_1 & a_2 & \cdots & a_{2r} \end{vmatrix} = 0.
\end{aligned}
\tag{3}
$$

On the other hand, by the classical Laplace expansion for determinants, it can be obtained that

$$
\begin{aligned}
&\begin{vmatrix} M & 0 & a_1 & a_2 & \cdots & a_{2r} \\ 0 & M & a_1 & a_2 & \cdots & a_{2r} \end{vmatrix} \\
&= \sum_{\substack{1 \le k_1 < \cdots < k_r \le 2r \\ k_{r+1} < \cdots < k_{2r}}} (-1)^{\frac{n(n+1)}{2} + k_1 + \cdots + k_r} \begin{vmatrix} M & a_{k_1} & a_{k_2} & \cdots & a_{k_r} \end{vmatrix} \\
&\qquad \cdot \begin{vmatrix} M & a_{k_{r+1}} & a_{k_{r+2}} & \cdots & a_{k_{2r}} \end{vmatrix}.
\end{aligned}
\tag{4}
$$

Comparing the above two equations, we have

$$
\sum_{\sigma} (-1)^{k_1 + \cdots + k_r} \begin{vmatrix} M & a_{k_1} & \cdots & a_{k_r} \end{vmatrix} \cdot \begin{vmatrix} M & a_{k_{r+1}} & \cdots & a_{k_{2r}} \end{vmatrix} = 0,
$$

where $\{k_1, k_2, \cdots, k_{2r}\}$ is a permutation of $\{1, 2, \cdots, 2r\}$ and σ is the permutation with $1 \le k_1 < \cdots < k_r \le 2r$ and $k_{r+1} < \cdots < k_{2r}$. □

Corollary 2.2. *Let M be a $n \times (n-2)$ matrix and a, b, c and d four n-order column vectors , then*

$$
\begin{aligned}
&\begin{vmatrix} M & a & b \end{vmatrix} \cdot \begin{vmatrix} M & c & d \end{vmatrix} - \begin{vmatrix} M & a & c \end{vmatrix} \cdot \begin{vmatrix} M & b & d \end{vmatrix} \\
&\qquad + \begin{vmatrix} M & a & d \end{vmatrix} \cdot \begin{vmatrix} M & b & c \end{vmatrix} = 0.
\end{aligned}
\tag{5}
$$

□

Remark 2.3. *The above equation* (5) *is the simplest case of the Plücker relations* [2] *which plays an important role in nonlinear dynamics and soliton theory due to Sato's theorem* [3, 4]. *It shows that many of the differential and difference equations in mathematical physics are merely disguised versions of the Plücker relations. For example, Sato* [3, 4] *first discovered that the KP equation in bilinear form*

$$
\left(D_x^4 - 4D_x D_t + 3D_y^2\right) f \cdot f = 0 \tag{6}
$$

was nothing but a Plücker relation, where the Hirota's bilinear operators D_t, D_x and D_y are defined by

$$
D_x^m D_t^n f \cdot g = \left(\partial_x - \partial_{x'}\right)^m \left(\partial_t - \partial_{t'}\right)^n f(x, t)\, g(x', t')\big|_{x'=x,\, t'=t}.
$$

Remark 2.4. *The other forms of Plücker relations have been widely applied to algebraic geometry. For example, a projective embedding of the Grassmann variety Gr(p, n) can be defined by the quadratic polynomial equations called "the Plücker relations" (to see* [5] *and its references).*

Remark 2.5. *The Plücker relations are also exactly relational to the Maya diagrams and Young diagrams* [2]*. It makes them play a primary role in Combinatorics, Lie theory and Representation Theory (to see* [6, 7] *and their references).*

3. Jacobi identities for determinants

Theorem 3.1. *Let $A = \left(a_{ij}\right)_{n\times n}$ be a n-order matrix. Denoted by $M_{k\,l}^{i\,j}$ the cofactor of the minor determinant $\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix}$, then*

$$M_{k\,l}^{i\,j}M_{s\,r}^{i\,j} - M_{k\,s}^{i\,j}M_{l\,r}^{i\,j} + M_{k\,r}^{i\,j}M_{l\,s}^{i\,j} = 0. \tag{7}$$

Proof. It is no less of generality to consider the case of $i < j$ and $k < l < s < r$. Denoted by M the $(n-2)\times(n-4)$ submatrix obtained by eliminating the i-th and j-th rows and the k-th, l-th, s-th and r-th columns from A. The four $(n-2)$-order column vectors obtained by eliminating the i-th and j-th components from the k-th, l-th, s-th and r-th column vectors in A are denoted by a, b, c and d respectively. Then it is easy to see that

$$\begin{aligned}
M_{k\,l}^{i\,j} &= (-1)^{k+l}\left|\begin{matrix} M & a & b \end{matrix}\right|; &\quad M_{s\,r}^{i\,j} &= (-1)^{s+r}\left|\begin{matrix} M & c & d \end{matrix}\right|;\\
M_{k\,s}^{i\,j} &= (-1)^{k+s}\left|\begin{matrix} M & a & c \end{matrix}\right|; &\quad M_{l\,r}^{i\,j} &= (-1)^{l+r}\left|\begin{matrix} M & b & d \end{matrix}\right|;\\
M_{k\,r}^{i\,j} &= (-1)^{k+r}\left|\begin{matrix} M & a & d \end{matrix}\right|; &\quad M_{l\,s}^{i\,j} &= (-1)^{l+s}\left|\begin{matrix} M & b & c \end{matrix}\right|.
\end{aligned}$$

Consequently, by employing the Plücker relation (5), one has

$$\begin{aligned}
&M_{k\,l}^{i\,j}M_{s\,r}^{i\,j} - M_{k\,s}^{i\,j}M_{l\,r}^{i\,j} + M_{k\,r}^{i\,j}M_{l\,s}^{i\,j}\\
=&(-1)^{k+l+r+s}\Big(\left|\begin{matrix} M & a & b \end{matrix}\right|\cdot\left|\begin{matrix} M & c & d \end{matrix}\right| - \left|\begin{matrix} M & a & c \end{matrix}\right|\cdot\left|\begin{matrix} M & b & d \end{matrix}\right|\\
&\qquad + \left|\begin{matrix} M & a & d \end{matrix}\right|\cdot\left|\begin{matrix} M & b & c \end{matrix}\right|\Big)\\
=&0.
\end{aligned}$$

□

Remark 3.2. *Note that only the indices are important in the equation* (7)*, so it can also be expressed as*

$$(k,\,l)\,(s,\,r) - (k,\,s)\,(l,\,r) + (k,\,r)\,(l,\,s) = 0,$$

which is nothing but a Plücker relation.

Theorem 3.3. *Let $A = \left(a_{ij}\right)_{n\times n}$ be a n-order matrix. Denoted by $M_{j_1\cdots j_r}^{i_1\cdots i_r}$ the cofactor of the minor determinant $\begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_r} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_r} \end{vmatrix}$. Choosing r rows and $2r$ columns from A, the according row and column indices are denoted by*

$i_1, \cdots, i_r$ and $j_1, \cdots, j_{2r}$ respectively. It might as well set that $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_{2r}$. Then

$$\sum_{\sigma} (-1)^{k_1+\cdots+k_r} M^{i_1 \cdots i_r}_{k_1 \cdots k_r} M^{i_1 \cdots i_r}_{k_{r+1} \cdots k_{2r}} = 0, \tag{8}$$

where $\{k_1, k_2, \cdots, k_{2r}\}$ is a permutation of $\{j_1, j_2, \cdots, j_{2r}\}$ and σ is the permutation with $j_1 \le k_1 < \cdots < k_r \le j_{2r}$ and $k_{r+1} < \cdots < k_{2r}$.

Proof. It is completely similar to the proof of the above theorem 3.1 by employing the equation (2). So here we omit it. $\square$

Theorem 3.4. (Jacobi identity [1, 2]**)** *Let $A = (a_{ij})_{n\times n}$ be a n-order matrix. Denoted by M_{ij} the cofactor of the matrix entry a_{ij} in A. The cofactor of the minor determinant $\begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix}$ is denoted by $M^{i\,j}_{k\,l}$, then*

$$M_{ii}M_{jj} - M_{ij}M_{ji} = M^{i\,j}_{i\,j} \det A, \qquad 1 \le i,\, j \le n. \tag{9}$$

Proof. It is no less generality to condider the special case with $i = 1$ and $j = 2$. Firstly, it is easy to see that

$$M_{1\,1} = a_{22} M^{1\,2}_{1\,2} + \sum_{k=3}^{n} (-1)^k a_{2k} M^{1\,2}_{2\,k}; \qquad M_{2\,2} = a_{11} M^{1\,2}_{1\,2} + \sum_{l=3}^{n} (-1)^l a_{1l} M^{1\,2}_{1\,l};$$

$$M_{1\,2} = a_{21} M^{1\,2}_{1\,2} + \sum_{k=3}^{n} (-1)^k a_{2k} M^{1\,2}_{1\,k}; \qquad M_{2\,1} = a_{12} M^{1\,2}_{1\,2} + \sum_{l=3}^{n} (-1)^l a_{1l} M^{1\,2}_{2\,l}.$$

Then

$$\begin{aligned}
&M_{11}M_{22} - M_{12}M_{21} \\
=&\big(M^{1\,2}_{1\,2}\big)^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + M^{1\,2}_{1\,2} \sum_{l=3}^{n} (-1)^{l+1} M^{1\,2}_{1\,l} \begin{vmatrix} a_{12} & a_{1l} \\ a_{22} & a_{2l} \end{vmatrix} \\
&\qquad + M^{1\,2}_{1\,2} \sum_{k=3}^{n} (-1)^k M^{1\,2}_{2\,k} \begin{vmatrix} a_{11} & a_{1k} \\ a_{21} & a_{2k} \end{vmatrix} + \sum_{k,\,l=3}^{n} (-1)^{k+l} M^{1\,2}_{1\,k} M^{1\,2}_{2\,l} \begin{vmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{vmatrix}.
\end{aligned}$$

On the other hand, by the equation (7), it can be obtained that

$$\begin{aligned}
&\sum_{k,\,l=3}^{n} (-1)^{k+l} M^{1\,2}_{1\,k} M^{1\,2}_{2\,l} \begin{vmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{vmatrix} \\
=& \sum_{3\le k<l\le n} (-1)^{k+l} M^{1\,2}_{1\,k} M^{1\,2}_{2\,l} \begin{vmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{vmatrix} + \sum_{3\le l<k\le n} (-1)^{k+l} M^{1\,2}_{1\,k} M^{1\,2}_{2\,l} \begin{vmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{vmatrix} \\
=& \sum_{3\le k<l\le n} (-1)^{k+l} \big(M^{1\,2}_{1\,k} M^{1\,2}_{2\,l} - M^{1\,2}_{1\,l} M^{1\,2}_{2\,k}\big) \begin{vmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{vmatrix} \\
=& \sum_{3\le k<l\le n} (-1)^{k+l} M^{1\,2}_{1\,2} M^{1\,2}_{k\,l} \begin{vmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{vmatrix}.
\end{aligned}$$

Therefore, by employing the Laplace expasion for determinants, one has

$$
\begin{aligned}
&M_{11}M_{22} - M_{12}M_{21} \\
=&M_{1\,2}^{1\,2}\Bigg[M_{1\,2}^{1\,2}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \sum_{l=3}^{n}(-1)^{l+1}M_{1\,l}^{1\,2}\begin{vmatrix} a_{12} & a_{1l} \\ a_{22} & a_{2l} \end{vmatrix} \\
&\qquad + \sum_{k=3}^{n}(-1)^{k}M_{2\,k}^{1\,2}\begin{vmatrix} a_{11} & a_{1k} \\ a_{21} & a_{2k} \end{vmatrix} + \sum_{3\le k<l\le n}(-1)^{k+l}M_{k\,l}^{1\,2}\begin{vmatrix} a_{1k} & a_{1l} \\ a_{2k} & a_{2l} \end{vmatrix}\Bigg] \\
=&M_{1\,2}^{1\,2}\det A.
\end{aligned}
$$

□

Remark 3.5. *The Jacobi identities play an important role in soliton theory. For example* [2]*, if the solutions to the KP equation or the Toda lattice equation are expressed as Grammian determinants, their bilinear equations are nothing but the Jacobi identities.*

Corollary 3.6. *Let* $A = \left(a_{ij}\right)_{2n\times 2n}$ *be a* $2n$*-order antisymmetric matrix, then* $\det A$ *is equal to a perfect square of a polynomial of its matrix entries* a_{ij} $(1 \le i,\, j \le n)$*.*

Proof. Following the above symbols, it is obvious that $M_{11} = M_{22} = 0$ and $M_{12} = -M_{21}$. Hence using the Jacobi identities, it can be obtained that

$$M_{1\,2}^{1\,2}\det A = \left(M_{12}\right)^2. \tag{10}$$

Note that $\det A = a_{12}^2$ when $n = 1$, and $M_{1\,2}^{1\,2}$ is the determinant of a $2(n-1)$-order antisymmetric matrix. Therefore, it can be deduced that $\det A$ is a perfect square of a polynomial of a_{ij} $(1 \le i,\, j \le n)$ by the recurrence relation (10). □

Remark 3.7. *Recalling the definition of a Pfaffian, the square of a* n*-order Pfaffian is equal to the determinant of a* $2n$*-order antisymmetric matrix. The above corollary in a sense ensures that a Pfaffian is well defined. On the other hand, by the Pfaffian expression for determinants, the terms of Jacobi identities* (9) *can be expressed as*

$$\det A = \left(1,\, 2,\, \cdots,\, n,\, n^*,\, \cdots,\, 2^*,\, 1^*\right); \tag{11}$$

$$M_{ii} = \left(1,\, \cdots,\, \hat{i},\, \cdots,\, n,\, n^*,\, \cdots,\, \hat{i}^*,\, \cdots,\, 1^*\right); \tag{12}$$

$$M_{ij} = \left(1,\, \cdots,\, \hat{i},\, \cdots,\, n,\, n^*,\, \cdots,\, \hat{j}^*,\, \cdots,\, 1^*\right); \tag{13}$$

$$M_{ij}^{ij} = \left(1,\, \cdots,\, \hat{i},\, \cdots,\, \hat{j},\, \cdots,\, n,\, n^*,\, \cdots,\, \hat{j}^*,\, \cdots,\, \hat{i}^*,\, \cdots,\, 1^*\right), \tag{14}$$

where the Pfaffian entries are defined by $(i,\, j) = (i^*,\, j^*) = 0$, $(i,\, j^*) = -(j^*,\, i) = a_{ij}$*. Another proof of the Pfaffian version of Jacobi identities can also be found in* [7]*. Since the Jacobi identities are exactly relational to the structrue of solutions for soliton equations, the Pfaffian should also be. In*

fact, recently, many scholars [8, 9, 10, 11, 12, 13] *have committed themselves to this domain.*

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